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An Existence Theory for Functional-Differential Equations and Functional-Differential Systems

E. K. IFANTIS

Department of Mathematics, University of Patras, Patras, Greece

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1. INTRODUCTION

In this paper we present a method for determining the existence of analytic solutions of linear functional-differential equations and functional-differential systems. This method, similar to one used by the author [5, 6] for the study of the Schrödinger equation, reduces the existence problem to the problem of finding the null space of a non-self-adjoint operator in an abstract separable Hilbert space.

The method is based on a factorization of the differential operator:

$$L : Lf(z) = \sum_{\iota=0}^{k} \varphi_\iota(z) \frac{d^{(k-\iota)} f(z)}{dz^{k-\iota}},$$

defined on an appropriate dense domain of the Hardy–Lebesgue space $H_2(\Delta)$, i.e., the Hilbert space of analytic functions in the unit disk. The functions $\varphi_\iota(z)$, $\iota = 0, 1, 2,..., k$, are assumed to be analytic in some neighborhood of the closed unit disk.

With respect to the equations, the general case, which we consider here, is the following:

$$(L + A) f(z) = 0, \qquad f(z) \in H_2(\Delta),$$

where A is any bounded operator on $H_2(\Delta)$.

In particular we consider the functional-differential equation:

$$Lf(z) + \sum_{\iota=1}^{\infty} a_\iota(z) f(q^\iota z) = 0, \qquad |q| \leqslant 1,$$

where $a_\iota(z)$, $\iota = 1, 2,..., k$ are analytic functions in some neighborhood of the closed unit disk. In the regular case $\varphi_0(z) = 1$ the main results generalize the classical existence theorem and previous results [4], obtained for $k = 1$ and $a_\iota(z)$ not functions but complex constants. In the singular case $\varphi_0(z) = z^\mu$,

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$\mu \leqslant k$ we obtain a generalization of the well known theorem of Perron [7]. With respect to the systems, the general case, which we consider here is the following:

$$z^D f'(z) = A_{\iota j} f(z),$$

where $f(z)$ is a vector function: $f(z) = (f_1(z), f_2(z),..., f_k(z))$ in the product space $H_2(\Delta)^k = H_2(\Delta) \times$ (k-times) $H_2(\Delta)$, D is the diagonal $(d_1, d_2,..., d_k)$ and $A_{\iota j}$ any bounded operator on $H_2(\Delta)^k$, i.e., any matrix of bounded operators on $H_2(\Delta)$.

We obtain for the above system the following theorem (Theorem 11). If trace $D = d$ and $k - d \geqslant 0$, then the system has at least $k - d$ linearly independent solutions in $H_2(\Delta)^k$ for every bounded operator $A_{\iota j}$. A corollary to this theorem generalizes some previous results obtained in Refs. [1–3].

Finally we consider as a typical example the following equation:

$$y'(x) = ay(\lambda x) + by(x), \qquad 0 \leqslant x < \infty, \tag{A}$$

which was studied in Ref. [10]. We obtain the well-known result that the problem associated with (A) and the condition $y(0) = 1$ is well posed if $\lambda < 1$. This result can be easily generalized for the following system:

$$f'(z) = (R + T)f(z), \qquad f(0) = 0,$$

where $f(z)$ is a vector function in $H_2(\Delta)^k$ and R, T are defined as follows: $Tf(z) = g(z) = (g_1(z), g_2(z),..., g_k(z))$, where:

$$g_\iota(z) = \sum_{j=1}^{k} b_{\iota j}(z) f_j(z) \qquad \text{and} \qquad R : Rf(z) = h(z) = (h_1(z),..., h_k(z)),$$

where:

$$h_i(z) = \sum_{j=1}^{k} a_{\iota j}(z) f_j(\mu_{\iota j} z), \qquad \text{and} \qquad | \mu_{\iota j} | \leqslant 1, \quad \iota, j = 1, 2,..., k.$$

The solution $f(z)$ is an entire vector function in case $b_{\iota j}(z)$ and $a_{\iota j}(z)$ are entire functions.

For $| \lambda | > 1$ the problem associated with (A) is reduced to an eigenvalue problem of a compact (non-self-adjoint) operator on $H_2(\Delta)$. The eigenvalues are exactly determined (Theorem 12) and we obtain for Eq. (A) the following result. In case $| \lambda | > 1$ Eq. (A) has analytic solutions only for $a = 0$ and $a = -b/\lambda^{k-1}$, $k = 1, 2, 3,...$. Moreover the solutions for $a \neq 0$ are polynomials of degree $k - 1$.

2. The Factorization of the Differential Operator L

Let $H_2(\Delta)$ denote the Hardy–Lebesgue space, consisting of all analytic functions $f(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$ in the open unit disk $\Delta = \{z: | z | < 1\}$, which satisfy the condition $\sum_{n=1}^{\infty} | a_n |^2 < \infty$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for an abstract separable Hilbert space H and let V be the unilateral shift operator on H ($V : Ve_n = e_{n+1}$, $n = 1, 2,...$).

Consider the eigenelements $f_z = \sum_{n=1}^{\infty} z^{n-1} e_n$, $z \in \Delta$ of V^*, the adjoint of V ($V^*e_n = e_{n-1}$ for $n \neq 1$, $V^*e_1 = 0$).

Every function $f(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$ in $H_2(\Delta)$ can be represented as follows:

$$f(z) = \langle f_z, f \rangle, \tag{1}$$

where $\langle\ ,\ \rangle$ is the scalar product in H, f is the element $f = \sum_{n=1}^{\infty} a_n^* e_n$ in H, and a_n^* is the complex conjugate of a_n. Of course if $f(z) = \langle f_z, f \rangle$, then $zf(z) = \langle f_z, Vf \rangle$ and

$$z^n f(z) = \langle f_z, V^n f \rangle, \qquad z \in \Delta, \tag{2}$$

because $V^* f_z = z f_z$.

Proposition 1.[1] *Assume that $\varphi(z) = \sum_{n=1}^{\infty} c_n z^{n-1}$ is analytic in some neighborhood of the closed unit disk $\bar{\Delta} = \{z: | z | \leqslant 1\}$. Then*

$$\varphi(V) = \lim_{n\to\infty} \sum_{k=1}^{n} c_k V^{k-1} = \sum_{n=1}^{\infty} c_n V^{n-1}$$

is a bounded operator on H and by the representation (1) *we have*:

$$\varphi(z) \cdot f(z) = \langle f_z, \varphi^*(V) f \rangle, \tag{3}$$

where $\varphi^*(V) = \sum_{n=1}^{\infty} c_n^* V^{n-1}$.

Proof. Since $\varphi(z)$ is analytic in some neighborhood of $\bar{\Delta}$ it converges absolutely for $| z | = 1$. That means that: $\forall \epsilon > 0$ there exists $N(\epsilon)$ such that

$$\sum_{k=m+1}^{n} | c_k | < \epsilon \qquad \forall n > m \geqslant N(\epsilon).$$

Let $\varphi_n(V) = \sum_{k=1}^{n} c_k V^{k-1}$. Then $\varphi_n(V) - \varphi_m(V) = \sum_{k=m+1}^{n} c_k V^{k-1}$ and since $\| V \| = 1$ it follows that:

$$\| \varphi_n(V) - \varphi_m(V) \| = \left| \sum_{k=m+1}^{n} c_k V^{k-1} \right| \leqslant \sum_{k=m+1}^{n} | c_k | < \epsilon \quad \text{for } n > m \geqslant N(\epsilon).$$

[1] The first assertion of Proposition 1 is a particular case of a general theorem concerning functions of bounded operators [11].

Thus the sequence of bounded operators $\varphi_n(V)$ converges uniformly to $\varphi(V)$, which is therefore a bounded operator. Relation (3) follows easily from (2) and the fact that $\lim(\varphi_n(V))^* = (\varphi(V))^* = \sum_{n=1}^{\infty} a_n{}^* V^{*n-1}$, $n \to \infty$.

Define the diagonal operator C_0 as follows:

$$C_0 e_n = n e_n , \qquad n = 1, 2, \ldots .$$

The operator C_0 is obviously defined on the dense linear manifold spanned by the basis $\{e_n\}_{n=1}^{\infty}$.

PROPOSITION 2. *The operator C_0 has a self-adjoint extension with discrete spectrum, i.e., the definition domain of C_0 can be extended to the range of the bounded operator B, which is defined as follows:*

$$Be_n = (1/n)\, e_n , \qquad n = 1, 2, \ldots .$$

Proof. First observe that B is self-adjoint and bounded. The inverse of B exists, and is defined on the range of B which is dense in H because B is self-adjoint and the zero is not an eigenvalue. Since $C_0 B = BC_0 = I$ (the identity operator on H) on the dense linear manifold spanned by the basis $\{e_n\}$ the inverse of B is an extension of C_0.

Let $\lambda \neq n$, $n = 1, 2, \ldots$. Then $(n - \lambda)^{-1}$ tends to zero and since $(C_0 - \lambda I)^{-1} e_n = (n - \lambda)^{-1} e_n$, the operator $(C_0 - \lambda I)^{-1}$ is bounded. Thus λ is not in the spectrum of C_0, which therefore consists of the points $n = 1, 2, \ldots$.

In the same way we can show that the definition domain of the operators $C_0{}^p$, $p = 2, 3, \ldots, k$ is extended to the range of the bounded operators $B^p, p = 2, 3, \ldots, k$.

PROPOSITION 3. *The range of B^p in H, $p = 1, 2, \ldots, k$, i.e., the definition domain of $C_0{}^p$ is isomorphic to the linear manifold in $H_2(\Delta)$ which consists of functions with derivatives up to order p in $H_2(\Delta)$.*

Proof. First observe that the definition domain of $C_0{}^p$, $p = 1, 2, \ldots, k$ is the same as the definition domain of $(C_0 V^*)^p$. In fact $f = \sum_{n=1}^{\infty} a_n^* e_n$ in the definition domain of $(C_0 V^*)^p$ means that $\sum_{n=1}^{\infty} (n-1)^2 (n-2)^2 \cdots (n-p)^2 \mid a_n \mid^2 < \infty$ and this means that f lies in the definition domain of

$$(C_0 - I) \cdot (C_0 - 2I) \cdots (C_0 - pI),$$

which is the same as the definition domain of $C_0{}^p$. Let $f(z) = \langle f_z , f \rangle$ and assume that f lies in the definition domain of $(C_0 V^*)^p$. Then:

$$\begin{aligned} &\langle f_z , (C_0 V^*) f \rangle = f'(z), \\ &\langle f_z , (C_0 V^*)^2 f \rangle = \langle f_z , C_0 V^* f \rangle' = f''(z), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \qquad\qquad (4) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\langle f_z , (C_0 V^*)^p f \rangle = f^{(p)}(z). \end{aligned}$$

Conversely let $f(z) = \langle f_z, f\rangle$ with f in H has a derivative up to order p, $p = 1, 2, \ldots, k$ in $H_2(\Delta)$. Then $\sum_{n=1}^{\infty} (n-1)^2(n-2)^2 \cdots (n-p)^2 \mid a_n \mid^2 < \infty$ which means that f lies in the definition domain of

$$(C_0 - I) \cdot (C_0 - 2I) \cdots (C_0 - pI).$$

Denote the definition domain of $C_0{}^p$, $p = 1, 2, \ldots, k$ by $D(C_0{}^p)$, $p = 1, 2, \ldots, k$. Then

$$C_0 V - V C_0 = V \tag{5}$$

in the sense that $(C_0 V - V C_0)f = Vf$ for every f in $D(C_0)$. Also:

$$V^* C_0 - C_0 V^* = V^* \tag{6}$$

and:

$$\begin{aligned} (C_0 - I)f &= V C_0 V^* f, & f &\in D(C_0), \\ (C_0 - I) \cdot (C_0 - 2I)f &= V^2 (C_0 V^*)^2 f, & f &\in D(C_0{}^2), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ (C_0 - I) \cdot (C_0 - 2I) \cdots (C_0 - kI)f &= V^k (C_0 V^*)^k f, & f &\in D(C_0{}^k). \end{aligned} \tag{7}$$

Now consider the differential equation:

$$Lf(z) = \sum_{\iota=0}^{k} \varphi_\iota(z) f^{(k-\iota)}(z) = 0 \tag{8}$$

and assume that the functions $\varphi_\iota(z)$, $j = 0, 1, 2, \ldots, k$ are analytic in some neighborhood of $\bar{\Delta}$. Thus the operators $\varphi_\iota(V)$, $\iota = 0, 1, 2, \ldots, k$ are bounded, according to the Proposition 1, and the operator

$$\tilde{L} = \sum_{\iota=0}^{k} \varphi_\iota{}^*(V)(C_0 V^*)^{k-\iota} \tag{9}$$

is defined on the dense linear manifold $D(C_0{}^k)$.

THEOREM 1. *$f(z) = \langle f_z, f\rangle$ with f in $D(C_0{}^k)$ is a solution of the Eq.* (8) *if and only if the element f belongs to the null space of the operator* (9).

The proof follows easily from (1), (3), (4), and from the fact that the set of elements f_z is complete in the sense that if f is orthogonal to f_z for every $z : \mid z \mid < 1$ then $f = 0$.

THEOREM 2. *The operator* (9) *has the following factorization*:

$$\tilde{L} = A(k) \cdot [\varphi_0{}^*(V)\, V^{*k} + K], \tag{10}$$

where $A(k)$ is a diagonal unbounded operator with a bounded inverse and K is compact.

Proof. From (5) we obtain $(C_0V^*)^k = (C_0(C_0 + I) \cdots (C_0 + (k-1)V^{*k}$. Observe that the diagonal operator $A(k) = C_0(C_0 + I) \cdots (C_0 + (k-1)I)$ has a compact inverse $A^{-1}(k)$. Thus the operator (9) can be written:

$$\tilde{L} = A(k) \cdot \left[A^{-1}(k)\, \varphi_0{}^*(V)\, A(k)\, V^{*k} + A^{-1}(k) \sum_{\iota=1}^{k} \varphi_\iota{}^*(V)\, A(k-\iota)\, V^{*k-1}\right]. \tag{11}$$

The term $A^{-1}(k)\, \varphi_k{}^*(V)$ is obviously compact because $A^{-1}(k)$ is compact and $\varphi_k{}^*(V)$ is bounded. Consider the term $A^{-1}(k)\, \varphi_\iota{}^*(V)\, A(k-\iota)$, $1 \leqslant \iota \leqslant k-1$ and let $\varphi_\iota{}^*(V) = b_0 + b_1V + b_2V^2 + \cdots$. Define the sequence of operators $S(m)$ as follows:

$$S(m) = b_0A^{-1}(k)\, A(k-\iota) + b_1A^{-1}(k)\, VA(k-\iota) + \cdots + b_mA^{-1}(k)\, V^mA(k-\iota).$$

The term $b_0A^{-1}(k)\, A(k-\iota)$, $\iota \neq 0$, is compact because it is a diagonal operator and its diagonal is a null sequence [8]. It is easy to see that each of the other terms of $S(m)$ is also a compact operator. Thus $S(m)$ is a compact operator. Therefore $A^{-1}(k)\, \varphi_\iota{}^*(V)\, A(k-\iota)$ is compact because it is the limit in the uniform topology of the sequence of compact operators $S(m)$. The conclusion is that the operator

$$K_1 = A^{-1}(k) \sum_{\iota=1}^{k} \varphi_i{}^*(V)\, A(k-\iota)\, V^{*k-\iota} \tag{12}$$

is compact. Similarly we can prove that the operator:

$$K_2 = A^{-1}(k)\, \varphi_0{}^*(V)\, A(k) - \varphi_0{}^*(V) \tag{13}$$

is also compact. From (11), (12), and (13) we obtain:

$$\tilde{L} = A(k)[(\varphi_0{}^*(V) + K_2)\, V^{*k} + K_1] \tag{14}$$

and from (14) we have by setting $K = K_1 + K_2V^{*k}$

$$\tilde{L} = A(k)T, \qquad \text{where} \quad T = \varphi_0{}^*(V) \cdot V^{*k} + K. \tag{15}$$

3. The Functional Operator $Rf(z) = \sum_{\iota=1}^{\infty} a_\iota(z)\, f(q^\iota z)$, $|q| \leqslant 1$

Assume that $a_\iota(z)$, $\iota = 1, 2, \ldots$, are analytic in some neighborhood of the closed unit disk $\bar{\Delta}$. Then according to the representation (1) the functions $a_\iota(z)$ are represented by the bounded operators $a_\iota{}^*(V)$ as the function $\varphi(z)$ in (3). Consider the operators $R_1, R_2, \ldots, R_k, \ldots$, which are defined on $H_2(\Delta)$ as follows:

$$
\begin{aligned}
R_1 f(z) &= f(qz), \qquad |q| \leqslant 1, \\
R_2 f(z) &= f(q^2 z) = R_1^2 f(z), \\
&\vdots \\
R_k f(z) &= f(q^k z) = R_1^k f(z). \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots
\end{aligned}
$$

By (1) we have

$$f(qz) = \langle f_{qz}, f \rangle = \langle \tilde{R}_1 f_z, f \rangle = \langle f_z, \tilde{R}_1^* f \rangle,$$

where $\tilde{R}_1$ is defined on H as follows:

$$\tilde{R}_1 e_n = q^{n-1} e_n, \qquad n = 1, 2, \ldots . \tag{16}$$

Since $|q| \leqslant 1$ the operator (16) is obviously a bounded operator (Unitary in case $|q| = 1$ and compact in case $|q| < 1$.) Thus the operator R (if it exists) is represented in H by the operator:

$$\tilde{R} = \sum_{\iota=1}^{\infty} a_\iota^*(V) \tilde{R}_\iota^*. \tag{17}$$

The operator (17) is not always defined on H. For instance, let $a_\iota^*(V) = V^{\iota-1}$, $\iota = 1, 2, \ldots$. Then

$$\tilde{R} e_1 = \sum_{\iota=1}^{\infty} V^{\iota-1} \tilde{R}_\iota^* e_1 = \sum_{\iota=1}^{\infty} V^{\iota-1} e_1 = \sum_{\iota=1}^{\infty} e_i \quad \text{and} \quad \| \tilde{R} e_1 \|^2 = 1 + 1 + \cdots.$$

As we shall see later (Theorem 5) it is interesting to know under what conditions $\tilde{R}$ is bounded. A condition in order that $\tilde{R}$ is bounded is given in the following theorem.

THEOREM 3. *If* $|q| \leqslant 1$ *and* $\| a_\iota^*(V) \| \leqslant \mu_\iota$ *with* $\sum_{\iota=1}^{\infty} \mu_\iota < \infty$ *then* $\tilde{R}$ *is bounded.*

Proof. The proof is easy. In this case the sequence of bounded operators $S(m) = \sum_{\iota=1}^{\infty} a_\iota^*(V) \tilde{R}_\iota$ converges uniformly to $\tilde{R}$ and $\tilde{R}$ is therefore bounded (Note that $\| \tilde{R}_\iota \| = \sup | q |^{(n-1)\cdot\iota} = 1$, $n = 1, 2, \ldots$.)

4. THE DIFFERENTIAL-FUNCTIONAL EQUATION: (NONSINGULAR CASE)

$$\frac{d^k f}{dz^k} + \varphi_1(z) \frac{d^{k-1} f(z)}{dz^{k-1}} + \cdots + \varphi_k(z) f(z) + \sum_{\iota=1}^{\infty} a_\iota(z) f(q^\iota z) = 0, \quad |q| \leqslant 1. \tag{18}$$

Equation (18) is described in $H_2(\Delta)$ by the operator $M = L + R$, which is by (1) represented in H by the operator $\tilde{M} = \tilde{L} + \tilde{R}$ with $\varphi_0{}^*(V) = I$. Thus if $\tilde{R}$ is bounded $\tilde{M}$ has according to Theorem 2 the factorization:

$$\tilde{M} = A(k) \cdot (V^{*k} + S), \tag{19}$$

where $S = A^{-1}(k)\tilde{R} + K$ is a compact operator. Also $f(z) = \langle f_z , f \rangle$ is a solution of Eq. (18) if and only if the element f belongs to the null space of the operator $\tilde{M}$ or (since $A(k)$ is invertible) if and only if the element f belongs to the null space of the operator

$$T = V^{*k} + S. \tag{20}$$

Thus we obtain:

THEOREM 4. *The number of different solutions of Eq.* (18) *in* $H_2(\Delta)$ *is the dimension of the null space of the operator* (20).

The proof follows from the fact that different solutions (linearly independent) correspond by the representations (1) to linearly independent elements in H.

THEOREM 5. *If* $\tilde{R}$ *is bounded then the null space of the operator* (20) *is* k*-dimensional and Eq.* (18) *has therefore* k *different solutions in* $H_2(\Delta)$.

Proof. We have: $T = V^{*k}(I + V^k S)$ because $V^*V = I$. Since the null space of V^{*k} is k-dimensional (it is spanned by the elements $e_1 , e_2 ,..., e_k$) we have only to prove that $I + V^k S$ is invertible: Let $(I + V^k S)f = 0$ with f in H. Then $\langle f, e_m \rangle = 0$, $m = 1, 2,..., k$. Also $(I + V^k S)f = 0$ implies that f belongs to the range of $A^{-1}(k)$, i.e., to the definition domain of $A(k)$ and

$$A(k)(V^{*k} + S)f = 0$$

or $(\tilde{L} + \tilde{R})f = 0$. Since $\langle f, e_m \rangle = 0$ for $m = 1, 2,..., k$ it is easy to see that scalar multiplication of the latter equation by e_1 gives: $\langle f, e_{k+1} \rangle = 0$. Consequently scalar multiplication by e_2 gives: $\langle f, e_{k+2} \rangle = 0$ and so on. Thus $f = 0$. That means that the point -1 is not an eigenvalue of the operator $V^k S$ and since $V^k S$ is compact the Fredholm alternative implies that $I + V^k S$ is invertible.

Remark. A linear operator A is said to be a Fredholm operator if its range is closed and both the null space and the corange are finite dimensional. That means that the homogeneous equation $Af = 0$ has a finite number of linearly independent solutions in H and the inhomogeneous equation $Af = g$ can be solved provided g satisfies a finite number of conditions. In fact g must be orthogonal to the corange of A. More precisely if f_ι , $\iota = 1, 2,..., n$ is a basis of the corange of A then the equation $Tf = g$ can be solved if and only if g satisfies the conditions: $\langle g, f_1 \rangle = \langle g, f_2 \rangle = \cdots = \langle g, f_n \rangle = 0$. The operator $T = V^{*k} + S$ is a Fredholm operator and its corange is trivial. We give below a simple proof for this statement. A necessary and sufficient condition in order that a bounded operator A is a Fredholm operator is the following: There exists a bounded operator B such that the operators $AB - I$ and $BA - I$ are compact [8].

Since $V^*V = I$ and $VV^* = I - P$, where P is the one-dimensional projection $Pf = \langle f, e_1 \rangle e_1$, $f \in H$ we have for the operator $T = V^{*k} + S$: $TV^k - I = SV^k$ and $V^kT - I = V^kS - P - VPV^* - \cdots - V^{k-1}PV^{*k-1}$. And since P and S are compact it follows that the operators $TV^k - I$ and $V^kT - I$ are also compact, i.e., T is a Fredholm operator. To show that the corange of T is trivial let us assume that f_0 belongs to the corange of T. Then $\langle Tf, f_0 \rangle = 0$ for every f in H, or: $\langle (V^{*k} + S)f, f_0 \rangle = \langle V^{*k}(I + V^kS)f, f_0 \rangle = \langle (I + V^kS)f, V^kf_0 \rangle = 0$. But $I + V^kS$ is invertible (see the proof of Theorem 5). Thus for $f = (I + V^kS)^{-1}V^kf_0$ we obtain: $\| V^kf_0 \|^2 = 0$, which implies that $V^kf_0 = 0$ and therefore $f_0 = 0$. Thus the corange of T is trivial. That means that the inhomogeneous equation $Tf = g$ can be solved for every g in H.

We can find the solutions of the inhomogeneous equation (19) as follows: Let $A(k)Tf = g$, g in H. Then $A(k)(Tf - A^{-1}(k)g) = 0$ or $Tf - A^{-1}(k)g = 0$ and $V^{*k}[(I + V^kS)f - V^kA^{-1}(k)g] = 0$, which implies

$$(I + V^kS)f - V^kA^{-1}(k)g = c_1e_1 + c_2e_2 + \cdots + c_ke_k,$$

where c_ι, $\iota = 1, 2, \ldots, k$ are arbitrary constants. Thus:

$$f = (I + V^kS)^{-1}(c_1e_1 + c_2e_2 + \cdots + c_ke_k) + (I + V^kS)^{-1}V^kA^{-1}(k)g.$$

THEOREM 6. *Let $\tilde{R}$ be bounded and let $1 \leqslant m < k$. Then Eq.* (18) *has $k - m$ different solutions in $H_2(\Delta)$, which satisfy the conditions $f(0) = f'(0) = \cdots = f^{(m-1)}(0) = 0$.*

Proof. $f(0) = 0$ means in the representation (1) that $\langle f, e_1 \rangle = 0$ and $f^{(m-1)}(0) = 0$ means that $\langle f, e_m \rangle = 0$, i.e., the elements in the null space of the operator (20) must be of the form V^mg. Since $V^{*k} \cdot V^m = V^{*k-m}$ it is necessary and sufficient to find the null space of the operator $V^{*k-m} + S \cdot V^m$, which is $(k - m)$-dimensional. A more general perturbation theorem of the operator L is the following.

THEOREM 7. *Let A be a bounded operator on $H_2(\Delta)$. Then the equation $(L + A)f(z) = 0$ has at least one nontrivial solution in $H_2(\Delta)$ which satisfies the conditions $f(0) = f'(0) = \cdots = f^{(k-2)}(0) = 0$.*

Proof. Let A be represented on H by the bounded operator $\tilde{A}$. Then the equation $(L + A)f(z) = 0$ has at least one nontrivial solution in $H_2(\Delta)$ if and only if the null space of the operator $\tilde{L} + \tilde{A}$ is nontrivial or due to the factorization of $\tilde{L}$ if and only if the null space of $V^{*k} + G$, where G is compact, is nontrivial. As in Theorem 6 due to the conditions of the theorem the problem consists in finding the null space of the operator $V^* + GV^{k-1}$, which is nontrivial. In fact $V^* + GV^{k-1} = V^*(I + VGV^{k-1})$ and Fredholm alternative implies that either 0 is an eigenvalue of $I + VGV^{*k-1}$ or it is invertible. In both cases the null space of $V^*(I + VGV^{k-1})$ is nontrivial.

5. The Singular Case $\varphi_0(z) = z^d$

Assume that $\varphi_0(z) = z^d$ where d is a nonnegative integer, i.e., the operator L is of the form:

$$Lf(z) = z^d f^{(k)}(z) + \sum_{\iota=1}^{k} \varphi_\iota(z) f^{(k-\iota)}(z). \tag{21}$$

The function $\varphi_\iota(z)$, $\iota = 1, 2,..., k$ are also analytic on $\bar{\Delta}$.

Theorem 8. *Let $k - d \geqslant 0$. Then for every bounded operator A in $H_2(\Delta)$ the equation*

$$(L + A) f(z) = 0 \tag{22}$$

has at least $k - d$ linearly independent solutions in $H_2(\Delta)$.

Proof. Let A be represented on H by the bounded operator $\tilde{A}$. Then

$$\tilde{L} + \tilde{A} = V^d(C_0V^*)^k + \sum_{\iota=1}^{k} \varphi_\iota{}^*(V)(C_0V^*)^{k-\iota} + \tilde{A}.$$

But $V^d(C_0V^*)^k = V^d(C_0V^*)^d$.

$$(C_0V^*)^{k-d} = (C_0 - I)(C_0 - 2I) \cdots (C_0 - dI)(C_0V^*)^{k-d},$$

because of (7), and $V^d(C_0V^*)^k = \Delta \cdot V^{*k-d} = (\Delta - aI)\, V^{*k-d} + aV^{*k-d}$ for some a not in the spectrum of

$$W = (C_0 - I) \cdot (C_0 - 2I) \cdots (C_0 - dI)\, C_0(C_0 + I) \cdots (C_0 + [k - d - 1]I).$$

Thus the inverse $(W - aI)^{-1}$ is a compact operator and as in Theorem 5 we obtain:

$$\tilde{L} + \tilde{A} = (W - aI) \cdot (V^{*k-d} + S), \tag{23}$$

where S is compact. Now the theorem follows easily from a general theorem concerning compact perturbation of Fredholm operators [9] (if T is Fredholm and S compact then $T + S$ is also Fredholm and

$$\dim(\ker T) - \dim(\ker T^*) = \dim(\ker(T + S)) - \dim(\ker(T^* + S^*))).$$

Since V^{*k-d} is Fredholm with $\dim(\ker V^{*k-d}) = k - d$ and since the null space of V^{k-d} is trivial we have:

$$\dim(\ker(V^{*k-d} + S)) = k - d + \dim(\ker(V^{k-d} + S^*)).$$

Remark. Theorem 8 is a generalization of the classical theorem of Perron [7] which was generalized by several authors (see Ref. [3] and the references presented there). Perron's theorem follows from Theorem 8 for $A = 0$.

In Theorem 8 the operator A can be any bounded operator on $H_2(\Delta)$. For instance A may be the operator R in Section 3 under the hypothesis which makes it bounded. Also A may be the operator

$$Nf(z) = \sum_{\iota=0}^{\mu} \beta_\iota(z) f^{(\mu-\iota)}(\mu_\iota z) \tag{24}$$

for $0 < |\mu_\iota| < 1$, $\iota = 1, 2,..., \mu < \infty$, because of the following theorem.

THEOREM 9. *Let $\beta_\iota(z)$, $\iota = 1, 2,..., \mu$ be analytic in some neighborhood of the closed unit disk. Then the operator* (24) *is densely defined in $H_2(\Delta)$ in case $|\mu_\iota| \leqslant 1$ and compact in case* $0 < |\mu_\iota| < 1$ *for every ι.*

Proof. Define the operators N_ι, $\iota = 1, 2,..., \mu$ as follows:

$$N_\iota e_n = \mu_\iota^{n-1} e_n, \qquad n = 1, 2,...,$$

The operator N_ι is unitary in the case when $|\mu_\iota| = 1$ and it is easy to see that the operator $C_0^{\mu-\iota} N_\iota$ is defined on the definition domain $D(C_0^{\mu-\iota})$, which is dense in H. In case $|\mu_\iota| < 1$ the operator $C_0^m N_\iota$, for every finite m, is defined on all of H because N_ι is bounded and its range contains the definition domain of C_0^m. Since $C_0^m N_\iota$ is a diagonal operator and since its diagonal $n^m \cdot \mu_\iota^{n-1}$ tends to zero it is a compact operator [8]. Thus the operator:

$$\begin{aligned}(C_0 V^*)^{\mu-\iota} N_\iota &= C_0(C_0 + I) \cdots (C_0 + \mu - \iota - 1)\, V^{*\mu-\iota} N_\iota \\ &= C_0(C_0 + I) \cdots (C_0 + \mu - \iota - 1)\, \mu_\iota^{\mu-\iota} N_\iota V^{*\mu-\iota}\end{aligned}$$

is compact.

From (4) we obtain:

$$\begin{aligned}\langle f_z, (C_0 V^*)^{\mu-\iota} N_\iota^* f\rangle &= \langle f_z, N_\iota^* f\rangle^{(\mu-\iota)} \\ &= \langle N_\iota f_z, f\rangle^{(\mu-\iota)} = \langle f_{\mu_\iota z}, f\rangle^{(\mu-\iota)} \\ &= f^{(\mu-\iota)}(\mu_\iota z).\end{aligned}$$

N is therefore represented on H by the compact operator

$$\tilde{N} = \sum_{\iota=1}^{\mu} \beta_\iota^*(V)(C_0 V^*)^{\mu-\iota} N_\iota^*.$$

6. Functional-Differential Systems

Let $H^k = H \times$ (k-times) H be the Hilbert space consisting of the k-vector elements $\bar{f} = (f_1, f_2, ..., f_k)$, $\bar{g} = (g_1, g_2, ..., g_k)$,... with f_ι and g_ι in H. The scalar product in H^k is defined as follows: $\langle \bar{f}, \bar{g} \rangle = \sum_{\iota=1}^{k} \langle f_\iota, g_\iota \rangle$.

Let $D = \operatorname{diag}(d_1, d_2, ..., d_k)$ with nonnegative integers d_ι.

Definition 1. The operator Ω_D is defined on H^k as follows:

$$\Omega_D \bar{f} = (V^{d_1-1} f_1, V^{d_2-1} f_2, ..., V^{d_k-1} f_k), \tag{25}$$

where $V^{d_\iota - 1}$ for $d_\iota = 1$ is the identity operator on H and for $d_\iota = 0$ is the operator V^*. For instance if $D = \operatorname{diag}(2, 1, 0, 0, ..., 0)$ and $\bar{f} = (f_1, f_2, ..., f_k)$ then

$$\Omega_D \bar{f} = (V f_1, f_2, V^* f_3, ..., V^* f_k).$$

The adjoint of Ω_D is defined as follows:

$$\Omega_D^* \bar{f} = (V^{*d_1-1} f_1, V^{*d_2-1} f_2, ..., V^{*d_k-1} f_k), \tag{26}$$

where $V^{*d_\iota - 1}$ for $d_\iota = 1$ is also the identity operator on H and for $d_\iota = 0$ is the operator V.

For instance $\Omega^*_{(2,1,0,...,0)} \bar{f} = (V^* f_1, f_2, V f_3, ..., V f_k)$. Obviously $\Omega_{(1,1,1,...,1)}$ is the identity operator on H^k. The null space of $\Omega^*_{(0,0,...,0)}$ is k dimensional. It is spanned by the elements $(e_1, 0, 0, ..., 0)$, $(0, e_1, ..., 0) \cdots (0, 0, ..., e_1)$. It is easy also to see that if $D = \operatorname{diag}(d_1, d_2, ..., d_k)$ and $d_\iota \geqslant 1$ then the null space of Ω_D is trivial.

Denote with $\ker A$ the null space of an operator A and with $\dim(\ker A)$ its dimension. We shall use later the following proposition

Proposition 4. *Let $D = \operatorname{diag}(d_1, d_2, ..., d_k)$. Let $d_{\iota_1} = d_{\iota_2} = \cdots = d_{\iota_m} = 0$, $d_{j_1} = d_{j_2} = \cdots = d_{j_\mu} = 1$ and let the other elements of the diagonal which are different from 0 and 1 be $\sigma_1, \sigma_2, ..., \sigma_{k-m-\mu}$. Then*

$$\begin{aligned} &\dim(\ker \Omega_D) = m, \\ &\dim(\ker \Omega_D^*) = \sigma + m + \mu - k, \quad \text{where } \sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_{k-m-\mu}. \end{aligned} \tag{27}$$

Proof. Let $D = \operatorname{diag}(..., d_{\iota_1}, ..., d_{j_1}, ..., \sigma_1, ..., d_{\iota_2}, ..., d_{j_2}, ..., \sigma_2, ...)$ and let $\bar{f} = (f_1, f_2, f_3, ..., f_k)$. Then

$$\Omega_D \bar{f} = (..., V^* f_{\iota_1}, ..., f_{j_1}, ..., V^{\sigma_1 - 1} f_{\sigma_1}, ..., V^* f_{\iota_2}, ..., f_{j_2}, ..., V^{\sigma_2 - 1} f_{\sigma_2}, ...). \tag{28}$$

Since for $\sigma_\iota - 1 \geqslant 0$ the null space of $V^{\sigma_\iota - 1}$ is trivial in H the vector element in (28) is the zero element in H^k if and only if

$$\bar{f} = (,..., c_1 e_1 ,..., 0,..., 0,..., c_2 e_1 ,..., 0,..., 0,...)$$

where c_1 and c_2 are arbitrary constants. This means that the null space of Ω_D is spanned by the m orthonormal elements in H^k:

$$(,..., e_1 ,..., 0, ..., 0,..., 0,..., 0,..., 0,...),$$
$$(,..., 0,..., 0,..., 0,..., e_1 ,..., 0,..., 0,...),... .$$

For the adjoint $\Omega_D{}^*$ we have:

$$\Omega_D{}^* \bar{f} = (,..., Vf_{\iota_1} ,..., f_{j_1} ,..., V^{*\sigma_1 - 1} f_{\sigma_1} ,..., Vf_{\iota_2} ,..., f_{j_2} ,..., V^{*\sigma_2 - 1} f_{\sigma_2} ,...).$$

Thus if $\bar{f}$ is in ker $\Omega_D{}^*$, $f_{\iota_1} = f_{j_1} = f_{\iota_2} = f_{j_2} = 0$ and

$$f_{\sigma_1} = c_1 e_1 + c_2 e_2 + \cdots + c_{\sigma_1 - 1} e_{\sigma_1 - 1} , \qquad f_{\sigma_2} = c_1 e_1 + c_2 e_2 + \cdots + c_{\sigma_2 - 1} e_{\sigma_2 - 1} ,... .$$

The null space of $\Omega_D{}^*$ is therefore spanned from

$$(\sigma_1 - 1) + (\sigma_2 - 1) + \cdots + (\sigma_{k-m-\mu} - 1) = \sigma - (k - m - \mu)$$

orthonormal elements.

DEFINITION 2. Denote by B the operator C_0^{-1} as it is defined in Proposition 2. We define the operator B on H^k as follows:

$$B\bar{f} = (Bf_1 , Bf_2 ,..., Bf_k). \tag{29}$$

B is obviously a compact operator on H^k because B is compact on H. Also if $\tilde{A}_{\iota j}$, $\iota, j = 1, 2,..., k$, is a matrix of bounded operators on H then $B\tilde{A}_{\iota j}$ is a matrix of compact operators on H, i.e., a compact operator on H^k. If $D =$ diag $(d_1 , d_2 ,..., d_k)$ we shall denote with D the operator on H^k which acts as follows:

$$D\bar{f} = (d_1 f_1 , d_2 f_2 ,..., d_k f_k).$$

Now we ask for solutions in $H_2(\Delta)^k$ of the following system:

$$z^D f'(z) = A_{\iota j} f(z) + g(z), \tag{30}$$

where $A_{\iota j}$ is a matrix of bounded operators on $H_2(\Delta)$ and $g(z)$, $f(z)$ are vector elements in $H_2(\Delta)^k$, i.e., $g(z) = (g_1(z), g_2(z),..., g_k(z))$ with $g_\iota(z)$ in $H_2(\Delta)$ and $f(z) = (f_1(z), f_2(z),..., f_k(z))$ with both $f_\iota(z)$ and $f_\iota'(z)$ in $H_2(\Delta)$.

THEOREM 10. *Let $A = (A_{\iota j})$, $\iota, j = 1, 2,..., k$ be a matrix of bounded operators* *on $H_2(\Delta)$ and let $A_{\iota j}$ be represented on H by the operators $\tilde{A}_{\iota j}$. Then Eq.* (30) *has a solution in $H_2(\Delta)^k$ if and only if the equation*

$$\Omega_D \bar{f} = (DB\Omega_D + B\tilde{A}) \cdot \bar{f} + B\bar{g}, \tag{31}$$

has a solution in H^k, where $\bar{g} = (g_1, g_2, \ldots, g_k)$ and $g_\iota(z) = \langle f_z, g_\iota \rangle, j = 1, 2, \ldots, k$.

Proof. The system (30) in $H_2(\Delta)$ is equivalent to the following system in H:

$$\begin{array}{l} V^{d_1}C_0V^*f_1 = \tilde{A}_{11}f_1 + \tilde{A}_{12}f_2 + \cdots + \tilde{A}_{1k}f_k + g_1, \\ V^{d_2}C_0V^*f_2 = \tilde{A}_{21}f_1 + \tilde{A}_{22}f_2 + \cdots + \tilde{A}_{2k}f_k + g_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ V^{d_k}C_0V^*f_k = \tilde{A}_{k1}f_1 + \tilde{A}_{k2}f_2 + \cdots + \tilde{A}_{kk}f_k + g_k, \end{array} \tag{32}$$

where $\bar{f} = (f_1, f_2, \ldots, f_k)$ with f_ι in the definition domain of C_0. Since $V^*V = I$ and $(C_0 - I)e_1 = 0$ we have: $VC_0V^* = C_0 - I$ in the definition domain of C_0 and $V^{d_\iota}C_0V^* = V^{d_\iota - 1}(C_0 - I)$, where $V^{d_\iota - 1}$ is the identity operator in case $d_\iota = 1$ and the operator V^* in case $d_\iota = 0$. Since

$$V^{d_\iota - 1}C_0 = (C_0 + [-d_\iota + 1]I)\, V^{d_\iota - 1} \qquad \text{for} \quad d_\iota \geqslant 1$$

and $V^*C_0 = (C_0 + I)V^*$ we have: $V^{d_\iota}C_0V^* = C_0V^{d_\iota - 1} - d_\iota V^{d_\iota - 1}$. Thus the system (32) is equivalent to the system:

$$\begin{array}{l} V^{d_1 - 1}f_1 = d_1BV^{d_1 - 1}f_1 + B\tilde{A}_{11}f_1 + \cdots + B\tilde{A}_{1k}f_k + Bg_1, \\ V^{d_2 - 1}f_2 = d_2BV^{d_2 - 1}f_2 + B\tilde{A}_{21}f_1 + \cdots + B\tilde{A}_{2k}f_k + Bg_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ V^{d_k - 1}f_k = d_kBV^{d_k - 1}f_k + B\tilde{A}_{k1}f_1 + \cdots + B\tilde{A}_{kk}f_k + Bg_k, \end{array}$$

which is exactly Eq. (31) in H^k.

Putting together Proposition 4 and Theorem 10 we obtain:

THEOREM 11. *Let $A = (A_{\iota j})$, $\iota, j = 1, 2, \ldots, k$, be a matrix of bounded operator on $H_2(\Delta)$. Let* trace *$D = d$ and $k - d \geqslant 0$. Then the system*

$$z^D f'(z) = A_{\iota j} f(z) \tag{33}$$

has at least $k - d$ linearly independent solutions in $H_2(\Delta)^k$.

Proof. The number of linearly independent solutions of the system (33) in $H_2(\Delta)^k$ is the dimension of the null space of the operator $\Omega = \Omega_D - DB\Omega_D - B\tilde{A}_{\iota j}$

in H^k. Since B is compact and Ω_D and $\tilde{A}_{\iota j}$ bounded the operators $DB\Omega_D$ and $B\tilde{A}$ are also compact. Thus as in Theorem 8 we obtain: $\dim(\ker \Omega) - \dim(\ker \Omega^*) = \dim(\ker \Omega_D) - \dim(\ker \Omega_D{}^*) = m - (\sigma + m + \mu - k) = k - (\sigma + \mu) = k - d$, because of (27). And $\dim(\ker \Omega) \geqslant k - d$.

Remark 1. Let $\mu_{\iota j}$ and $\lambda_{\iota j}$ be complex numbers with $|\mu_{\iota j}| \leqslant 1$ and $0 < |\lambda_{\iota j}| < 1$, $\iota, j = 1, 2,..., k$, and let $a_{\iota j}(z)$, $\beta_{\iota j}(z)$, $\iota, j = 1, 2,..., k$, be analytic functions on the closed unit disk. Then as we indicated in Section 3 the operators:

$$R_{\iota j} : R_{\iota j} f(z) = a_{\iota j}(z) f(\mu_{\iota j} z), \qquad f(z) \in H_2(\Delta), \quad \iota, j = 1, 2,..., k. \tag{34}$$

are bounded on $H_2(\Delta)$ for $|\mu_{\iota j}| \leqslant 1$ and compact for $0 < |\mu_{\iota j}| < 1$. Also due to the Theorem 9 the operators

$$N_{\iota j} : N_{\iota j} f(z) = \beta_{\iota j}(z) f'(\lambda_{\iota j} z), \tag{35}$$

are compact on $H_2(\Delta)$ for $0 < |\lambda_{\iota j}| < 1$.

Now let $f(z) = (f_1(z), f_2(z),..., f_k(z))$ and $g(z) = (g_1(z), g_2(z),..., g_k(z))$ be vector functions in $H_2(\Delta)^k$. Define the operators R and N as follows:

$$Rf(z) = g(z),$$

where

$$g_\iota(z) = \sum_{j=1}^{k} a_{\iota j}(z) f_j(\mu_{\iota j} \cdot z) = \sum_{j=1}^{k} R_{\iota j} f_j(z) \tag{36}$$

and

$$Nf(z) = g(z),$$

where

$$g_\iota(z) = \sum_{j=1}^{k} \beta_{\iota j}(z) f_j'(\lambda_{\iota j} \cdot z) = \sum_{j=1}^{k} N_{\iota j} f_j(z). \tag{37}$$

Since every operator $R_{\iota j}$ and $N_{\iota j}$ is bounded and compact, respectively, and since k is finite we can easily see that the operator (36) is bounded and the operator (37) compact on $H_2(\Delta)^k$. We note also that the matrix $A = (A_{\iota j})$ may be a matrix T of analytic functions $b_{\iota j}(z)$ on the closed unit disk, defined as:

$$\begin{aligned} Tf(z) &= g(z), \\ g_\iota(z) &= \sum_{j=1}^{k} b_{\iota j}(z) f_j(z). \end{aligned} \tag{38}$$

Thus as corollary to Theorem 11 we obtain the following result:

COROLLARY. *Let* T, N, R *be the operators* (38), (37), *and* (36), *respectively, and let* trace $D = d$ *with* $k - d \geqslant 0$. *Then the system*

$$z^D f'(z) = (T + R + N) f(z) \tag{39}$$

has at least $k - d$ *linearly independent solutions in* $H_2(\Delta)^k$.

This corollary generalizes a result obtained in Ref. [2] for

$$\mu_{ij} = \lambda_{ij} = a \quad \text{and} \quad 0 < |a| < 1.$$

Remark 2. In Ref. [1], where a similar result has been proved, the operators (36), (37), and (38) are defined for $\mu_{ij} = \lambda_{ij} = \lambda$ and $0 < |\lambda| < 1$ on a Banach space of analytic vector functions. On that space the above operators are compact. According to Theorem 11 we do not need here the compactness but the boundedness of the operators (36), (37), and (38).

7. A Typical Example

As a typical example we consider the following functional-differential equation:

$$y'(x) = ay(\lambda x) + by(x). \tag{40}$$

This equation was studied in Ref. [10] for $0 \leqslant x < \infty$; a, a complex constant; b, a real constant; and λ, a nonnegative constant. In Ref. [10] was proved that the problem associated with Eq. (40) and the condition

$$y(0) = 1 \tag{41}$$

is well posed if $\lambda < 1$, but not if $\lambda > 1$.

We can see the first assertion for $|\lambda| \leqslant 1$ as follows: We ask for entire solutions of Eq. (40). For this we set $x = rz$, $0 < r < \infty$, $y(rz) = f(z)$, and ask for solutions of the equation

$$(1/r)(df/dz) = af(\lambda z) + bf(z) \tag{42}$$

in the space $H_2(\Delta)$ which satisfy the condition $f(0) = 1$. According to (1), (4), and (17), Eq. (42) with the condition $f(0) = 1$ is equivalent to the following equation in H

$$C_0 V^* f = (ra^* R^* + rb^*) f \tag{43}$$

and the condition

$$\langle f, e_1 \rangle = 1, \tag{44}$$

where the operator R^* is defined on H as follows:

$$R^* e_n = \lambda^{n-1} e_n , \qquad n = 1, 2, \ldots . \tag{45}$$

We set $C_0^{-1} = B$ and obtain from (43): $C_0(V^* - a^*rBR^* - b^*rB)f = 0$ and $(V^* - a^*rBR^* - b^*rB)f = 0$ which is equivalent to the following:

$$(I - a^*rVBR^* - b^*rVB)f = ce_1 . \tag{46}$$

Taking into account the condition (44) we obtain from (46): $c = 1$. Thus

$$(I - a^*rVBR^* - b^*rVB)f = e_1 . \tag{47}$$

But the point 1 is not an eigenvalue of the compact operator $a^*rVBR^* + b^*rVB$, because $f = (a^*rVBR^* + b^*rVB)f$ implies $\langle f, e_1\rangle = \langle f, e_2\rangle = \cdots = 0$, so the Fredholm alternative implies that the operator acting on the left of (47) is invertible. Thus Eq. (47) has a unique solution in H.

Therefore Eq. (40) has a unique solution in $H_2(\Delta)$; $y(x) = y(rz)$, which is an entire function because it is analytic in $| z | < 1$ for every finite r.

The case $| \lambda | > 1$. Assume that Eq. (40) has for $| \lambda | > 1$ a solution $y(x)$, which is analytic in some neighborhood of zero. Let r be the radius of convergence of $y(x)$. We set $\lambda x = \rho z$, where $0 < \rho < r | \lambda |$. So $f(z) = y(\rho z)$ has a radius of convergence greater than unity and therefore belongs to $H_2(\Delta)$. Moreover $f(z)$ must satisfy the equation:

$$\frac{1}{q\rho} \cdot \frac{df(qz)}{dz} = af(z) + bf(qz), \tag{48}$$

where $q = 1/\lambda$ and $| q | < 1$. Thus in order to obtain analytic solutions of Eq. (40) it is necessary and sufficient to ask for solutions of Eq. (48) in the space $H_2(\Delta)$, for every finite ρ.

Equation (48) is represented in H as follows:

$$(1/q^*\rho)\, C_0 V^* R^* f = a^* f + b^* R^* f, \qquad f \in H, \tag{49}$$

where the operator R is defined as:

$$Re_n = q^{n-1} e_n , \qquad n = 1, 2, \ldots . \tag{50}$$

Since $| q | < 1$ the operator (50) is compact. The operator $C_0 V^* R^*$ is also compact, because of Theorem 9. So we have an eigenvalue problem of the compact operator:

$$Q^* = C_0 V^* R^* - q^* \rho b^* R^*, \tag{51}$$

$$Q^* f = E^* f, \qquad E^* = a^* q^* \rho. \tag{52}$$

Thus solutions exist only for a denumerable set of values of E^*.

We observe that $E = 0$ is an eigenvalue of the operator (51). We see this easily because for $a = 0$ Eq. (40) is trivial. We shall prove now the following theorem:

THEOREM 12. *The eigenvalues of Q^* which ate different from zero are precisely the values:*

$$E_k^* = -b^* \rho q^{*k}, \qquad k = 1, 2, 3, \ldots . \tag{53}$$

No multiplicity exists and the corresponding eigenelements f_k are of the form $f_k = \sum_{\iota=1}^{k} a_\iota e_\iota$.

Proof. First we prove the second assertion of the theorem. Let $Q^* f_k = -b^* \rho q^{*k} f_k$ for some k. Then $\langle Q^* f_k, e_k\rangle = \langle C_0 V^* R^* f_k, e_k\rangle - q^* \rho b^* \langle R^* f_k, e_k\rangle = k \cdot q^{*k}\langle f_k, e_{k+1}\rangle - q^{*k} \rho b^* \langle f_k, e_k\rangle = -b^* \rho q^{*k}\langle f_k, e_k\rangle$. This gives $\langle f_k, e_{k+1}\rangle = 0$. Consequently we find $\langle f_k, e_{k+2}\rangle = 0$ and so on. On the other hand if we normalize the elements f_k by setting $\langle f_k, e_k\rangle = 1$, then the components $\langle f_k, e_{k-1}\rangle \cdots \langle f_k, e_1\rangle$ are uniquely determined. To prove the first assertion we consider the operator Q, the adjoint of Q^*. We shall prove that its eigenvalues are precisely the values:

$$E_k = -b \rho q^k, \qquad k = 1, 2, \ldots, \tag{54}$$

which implies that the eigenvalues of Q^* are the values (53). Let $E \neq E_k$, $k = 1, 2, \ldots$, and let $Qg = Eg$. Then it is easy to see that

$$\langle g, e_1\rangle = \langle g, e_2\rangle = \cdots = 0,$$

i.e., $g = 0$. That means that the eigenvalues of Q are included in the set of values (54). It is also easy to verify that the values (54) are all eigenvalues of Q. In fact the components of the element g_k in the equation $Qg_k = E_k g_k$, apart from a normalization factor, are uniquely determined for $n = k + 1, k + 2, \ldots$. Since $\langle g_k, e_n\rangle / \langle g_k, e_{n-1}\rangle = (n-1)\, q^{n-k-1} / b\rho(q^{n-k} - 1)$ tends to zero for $n \to \infty$ it follows that g_k belongs to H.

From (53) and (52) we have $a = -bq^{k-1}$ $k = 1, 2, \ldots$. Thus we obtain the following corollary:

COROLLARY. *Equation* (40) *for* $|\lambda| > 1$ *has analytic solutions only for* $a = 0$ *and* $a = -b/\lambda^{k-1}$, $k = 1, 2, \ldots$. *Moreover the solutions for* $a \neq 0$ *are polynomials of degree* $k - 1$, $k = 1, 2, \ldots$.

Proof. For the eigenelements of Q^* we have: $f_k = a_1 e_1 + a_2 e_2 + \cdots + a_k e_k$. Thus $f(z) = \langle f_z, f_k\rangle = a_1^* + a_2^* z + \cdots + a_k^* z^{k-1}$.

Remark. The first result obtained for Eq. (40) can be easily generalized for the system:

$$f'(z) = (R + T) f(z), \qquad f(0) = 1, \tag{55}$$

where R and T are the operators (36) and (38), and $|\mu_{ij}| \leqslant 1$. In fact Eq. (55) is represented in H^k by the equation: $C_0 V^* \bar{f} = (\tilde{R} + \tilde{T})\bar{f}$ from which we obtain $\bar{f} = VB(\tilde{R} + \tilde{T})\bar{f} + \bar{c}e_1$.

The condition $f(0) = 1$ implies $\bar{c} = (1, 1, \ldots, 1)$ and it is easy to see that the null space of the compact operator $VB(\tilde{R} + \tilde{T})$ is trivial. Thus in the same way as for Eq. (43) we obtain one and only one solution $\bar{f}$ in H^k and therefore one and only one solution of (55) in $H_2(\Delta)^k$. Moreover we note that if the functions $b_{ij}(z)$ and $a_{ij}(z)$ in (38) and (36) are entire functions then the components of the vector solution $f(z)$ are also entire functions, because the system (55) after the transformation $z = rw$ has also a unique solution in $H_2(\Delta)^k$ for every finite r.

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